

Equi-entangled bases in arbitrary dimensions

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Abstract

For the space of two identical systems of arbitrary dimensions, we introduce a continuous family of bases with the following properties: i) the bases are orthonormal, ii) in each basis, all the states have the same values of entanglement, and iii) they continuously interpolate between the product basis and the maximally entangled basis. The states thus constructed may find applications in many areas related to quantum information science including quantum cryptography, optimal Bell tests and investigation of enhancement of channel capacity due to entanglement.

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1 Introduction

In quantum computation and quantum information theory [1, 2] and many of the related fields the well known Bell basis finds numerous applications. This basis consists of four ortho-normal maximally entangled states of two qubits, denoted by:

$$|\phi^\pm\rangle = \frac{1}{\sqrt{2}}(|0,0\rangle \pm |1,1\rangle), \quad (1)$$

$$|\psi^\pm\rangle = \frac{1}{\sqrt{2}}(|0,1\rangle \pm |1,0\rangle). \quad (2)$$

The analogs of these states have been defined for d -dimensional systems, or qudits in the following form

$$|\psi_{m,n}\rangle := \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} \xi^{jn} |j, j+m\rangle, \quad m, n = 0 \cdots d-1. \quad (3)$$

where $\xi := e^{\frac{2i\pi}{d}}$ and $\{|0\rangle, \dots, |d-1\rangle\}$ is an orthonormal basis for the space of one qudit. The states (3) are maximally entangled and are mutually orthogonal. These states have found applications in almost every generalization of quantum algorithms and protocols and investigations of entanglement in higher dimensional systems [3, 4, 5, 6, 7, 8, 9, 11, 12].

The present paper poses a simple question.

Can we construct a basis for the space of two qudits which has the following properties:

- 1- It is an orthogonal basis,
- 2- all the states have the same value of entanglement, and
- 3- by varying a set of parameters, the basis while retaining the above two properties, continuously changes from a product basis to a maximally entangled basis?

We call such a basis an equi-entangled orthogonal basis or an interpolating basis. This question has a simple and well-known answer for qubits, namely

$$|\phi^+(\theta)\rangle = \cos \theta |0,0\rangle + \sin \theta |1,1\rangle, \quad (4)$$

$$|\phi^-(\theta)\rangle = \sin \theta |0,0\rangle - \cos \theta |1,1\rangle, \quad (5)$$

$$|\psi^+(\theta)\rangle = \cos \theta |0,1\rangle + \sin \theta |1,0\rangle, \quad (6)$$

$$|\psi^-(\theta)\rangle = \sin \theta |0,1\rangle - \cos \theta |1,0\rangle. \quad (7)$$

This basis continuously connects a product basis (for $\theta = 0$) to a maximally entangled basis (for $\theta = \frac{\pi}{4}$).

We should stress that it is quite trivial to construct one single state which interpolates between a product state and a maximally entangled state. What we want is an orthonormal basis which connects a product basis to a maximally entangled basis. Moreover we require that all along the way

each of the basis states has the same value of entanglement.

To our knowledge such a construction has not been generalized to higher dimensions and it is our aim in this paper to obtain such an interpolating basis in higher dimensions. This construction turns out to be quite non-trivial, since it requires the solution of simultaneous non-linear equations the number of which increase with dimension.

Before proceeding we would like to point out some probable applications of these kinds of states.

In an interesting recent development it has been shown that optimal Bell tests do not require maximally entangled states. In fact as stated in [13], it has often been assumed that generalized Bell states of the form (3) represent the most nonlocal quantum states while in [13], it is shown that optimal states for tests of non-locality are not maximally entangled except in the case of qubits indicating that the role of dimension is more subtle than previously supposed. Therefore having a set of mutually orthogonal states, all with the same value of entanglement, may facilitate the search for optimal states that test non-locality in arbitrary dimensions. In effect having such an equi-entangled basis generalizes the concept of Bell state measurement so that the measurement projects the measured states into mutually orthogonal states with a prescribed and tunable value of entanglement.

There is an important question in quantum information theory: is the classical capacity of quantum channels still additive if we allow encoding of input data into entangled states? Although there are clues that in channels without memory [14, 15] (where there is no correlation between two consecutive uses of the channel), entanglement does not enhance the capacity, for channels with memory there are clues that this is not the case [16, 17]. Depending on the value of correlations and the type of channel, one may want to encode the data (the alphabet or the strings) into mutually orthogonal and hence distinguishable quantum states with a prescribed and not necessarily maximal value of entanglement. The analysis of such a channel will then be made much easier if all the states have the same value of entanglement. In other words such states make it possible to have mixed states of arbitrary entanglement by taking

$$\rho = \sum_i P_i |\psi_i\rangle\langle\psi_i|,$$

where $\{|\psi_i\rangle\}$ is an equi-entangled orthonormal basis.

Other areas in which such states may find applications are quantum cryptography, quantum dense coding and quantum teleportation [1]. In all these areas Bell states and Bell state measurements play an essential role and all of them have been generalized to arbitrary dimensions. A generalized Bell state measurement in which the degree of entanglement of the output states can be tuned to any arbitrary value, may someday find applications in these areas.

Presumably these are not the only applications that one may envisage for these kinds of states.

The structure of this paper is as follows: In section 2 we present the general construction of these states in arbitrary dimensions and obtain the system of quadratic equations which should be solved explicitly in order to find the concrete form of the states. In sections 3 and 4 we present explicit solutions of the equations in three and four dimensions. Finally in section 5 we obtain the general

solution of the quadratic system of equations in arbitrary dimensions.

2 A method of construction

Here we present a method for construction of equi-entangled bases. Consider the single state

$$|\psi_{0,0}\rangle := \sum_{i=0}^{d-1} a_i |i, i\rangle, \quad (8)$$

and the unitary shift operator

$$S := \sum_{i=0}^{d-1} |i+1\rangle\langle i|. \quad (9)$$

For $m, n = 0, 1, \dots, d-1$, define d^2 states of the form

$$|\psi_{m,n}\rangle = S^m \otimes S^{m+n} |\psi_{00}\rangle = \sum_{i=0}^{d-1} a_i |i+m, i+m+n\rangle. \quad (10)$$

All these states have the same value of entanglement as the state $|\psi_{0,0}\rangle$, since all of them are obtained from $|\psi_{0,0}\rangle$ by bi-local unitary operators. The value of entanglement is obtained by using von Neumann entropy

$$E(|\psi_{m,n}\rangle) = E(|\psi_{00}\rangle) = - \sum_{i=0}^{d-1} |a_i|^2 \log_d |a_i|^2. \quad (11)$$

Note that we have taken the logarithms in base d so that a maximally entangled state has a value of entanglement equal to unity in all dimensions.

We now demand that they be ortho-normal. This leads to the following set of equations:

$$\sum_{i=0}^{d-1} \overline{a_i} a_{i+m} = \delta_{m,0} \quad m = 0, \dots, d-1. \quad (12)$$

This is a set of d quadratic equations for the complex coefficients a_i . The solution of this set of equations gives the final form of the state $|\psi_{00}\rangle$ and hence all the state vectors of the basis.

A natural question arises as to whether this is the only possible construction for obtaining such bases. We have tried many other constructions and have found that the ansatz presented in this section gives the most elegant or even inevitable solution. In fact the most general way of constructing d^2 equi-entangled states from $|\psi_{00}\rangle$ is to define the states as $|\psi_{mn}\rangle = U_m \otimes V_n |\psi_{00}\rangle$ where $\{U_m\}$ and $\{V_n\}$ are a set of $2d$ unitary operators. However orthogonality puts a stringent requirement on these operators in the form

$$\langle \psi_{00} | U_m^\dagger U_p \otimes V_n^\dagger V_q | \psi_{00} \rangle = \delta_{m,p} \delta_{n,q}. \quad (13)$$

One can simplify the situation by taking $U_m = U^m$ and $V_n = V^n$, hence searching for two unitary operators U and V with the property that $\langle \psi_{00} | U^m \otimes V^n | \psi_{00} \rangle = \delta_{m,0} \delta_{n,0}$. Calculations then show that the choice $U = V = S$ where S is the shift operator defined above gives the most elegant solution.

Despite this we refrain to claim that our construction is a general one.

Before proceeding to the general solution, for concreteness, we consider the cases of three and four dimensional spaces separately.

3 Equi-entangled basis for three dimensional spaces (qutrits)

3.1 Case a: Real vector space

If we assume that the basis vector $|\psi_{00}\rangle$ is real, all the other vectors will be real and we can construct a basis for a real vector space. In this case equations (12) reduce to the following two equations

$$\begin{aligned} a_0^2 + a_1^2 + a_2^2 &= 1 \\ a_0 a_1 + a_1 a_2 + a_2 a_0 &= 0. \end{aligned} \tag{14}$$

Parameterizing the variables in the first equation by polar coordinates

$$a_0 = \sin \theta \cos \phi, \quad a_1 = \sin \theta \sin \phi, \quad a_2 = \cos \theta, \tag{15}$$

and inserting these into the second equation we obtain a relation between the parameters, namely

$$\tan \theta = -\frac{\sin \phi + \cos \phi}{\sin \phi \cos \phi}. \tag{16}$$

Taking this relation into account we find the final form of the coefficients:

$$\begin{aligned} a_0 &= \frac{(\sin \phi + \cos \phi) \cos \phi}{1 + \sin \phi \cos \phi} \\ a_1 &= \frac{(\sin \phi + \cos \phi) \sin \phi}{1 + \sin \phi \cos \phi} \\ a_2 &= -\frac{(\sin \phi \cos \phi)}{1 + \sin \phi \cos \phi}. \end{aligned} \tag{17}$$

Thus we find a single parameter family of interpolating basis. For $\phi = 0$, we find $|\psi_{00}\rangle = |0, 0\rangle$ and hence the basis becomes disentangled. Using equation (11) we can calculate the entanglement. It is plotted in figure (1) as a function of ϕ .

The maximum entanglement is approximately equal to 0.87. This shows that in a real three dimensional space, we can not interpolate to a maximally entangled basis at least by the construction presented in this paper. However an interpolating basis exists in a complex three dimensional space as we show in the next subsection.

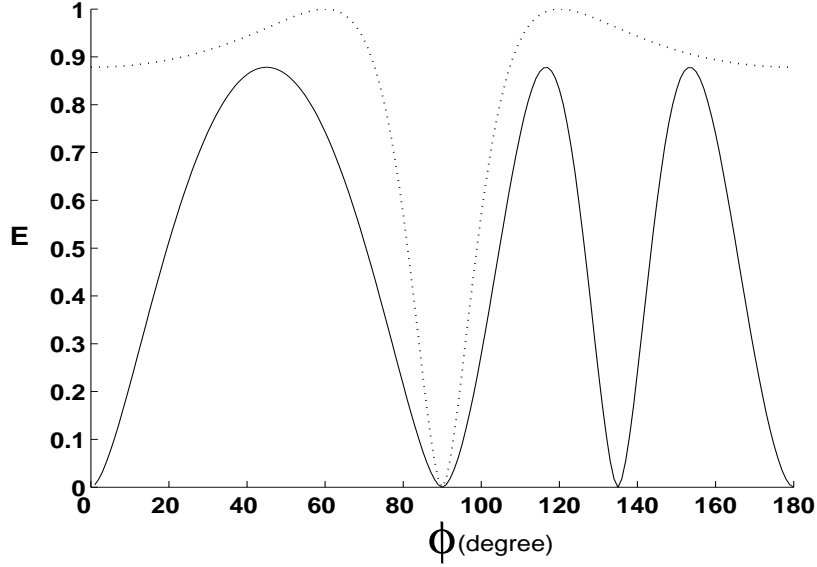


Figure 1: a) The entanglement of the basis (17) as a function of ϕ (in degree) (—). b) The entanglement of the basis (20) as a function of ϕ (in degree) (.....). The entanglement is dimensionless.

3.2 Case b: Complex vector space

Again in this case equations (12) reduce to two equations

$$\begin{aligned} |a_0|^2 + |a_1|^2 + |a_2|^2 &= 1 \\ \overline{a_0}a_1 + \overline{a_1}a_2 + \overline{a_2}a_0 &= 0. \end{aligned} \quad (18)$$

A solution for this set of equations leads to the following state $|\psi_{00}\rangle$

$$|\psi_{00}\rangle = \frac{1}{\sqrt{1+8\cos^2\phi}} (2\cos\phi|0,0\rangle - e^{i\phi}|1,1\rangle + 2\cos\phi|2,2\rangle). \quad (19)$$

The reader can read the coefficients from the above state and can check that equations (18) are satisfied. For convenience we list here the nine states of the basis, with $N := \frac{1}{\sqrt{1+8\cos^2\phi}}$, they read

$$\begin{aligned} |\psi_{00}\rangle &= N(2\cos\phi|0,0\rangle - e^{i\phi}|1,1\rangle + 2\cos\phi|2,2\rangle) \\ |\psi_{01}\rangle &= N(2\cos\phi|0,1\rangle - e^{i\phi}|1,2\rangle + 2\cos\phi|2,0\rangle) \\ |\psi_{02}\rangle &= N(2\cos\phi|0,2\rangle - e^{i\phi}|1,0\rangle + 2\cos\phi|2,1\rangle) \\ |\psi_{10}\rangle &= N(2\cos\phi|1,1\rangle - e^{i\phi}|2,2\rangle + 2\cos\phi|0,0\rangle) \\ |\psi_{11}\rangle &= N(2\cos\phi|1,2\rangle - e^{i\phi}|2,0\rangle + 2\cos\phi|0,1\rangle) \end{aligned}$$

$$\begin{aligned}
|\psi_{12}\rangle &= N(2 \cos \phi |1, 0\rangle - e^{i\phi} |2, 1\rangle + 2 \cos \phi |0, 2\rangle) \\
|\psi_{20}\rangle &= N(2 \cos \phi |2, 2\rangle - e^{i\phi} |0, 0\rangle + 2 \cos \phi |1, 1\rangle) \\
|\psi_{21}\rangle &= N(2 \cos \phi |2, 0\rangle - e^{i\phi} |0, 1\rangle + 2 \cos \phi |1, 2\rangle) \\
|\psi_{22}\rangle &= N(2 \cos \phi |2, 1\rangle - e^{i\phi} |0, 2\rangle + 2 \cos \phi |1, 0\rangle).
\end{aligned} \tag{20}$$

The reader can easily check that this basis is orthogonal. Moreover this basis interpolates between a product basis for $\phi = \frac{\pi}{2}$ and a maximally entangled basis for $\phi = \frac{\pi}{3}$. The entanglement of this basis is plotted as a function of ϕ in figure (1).

4 Equi-entangled basis for four dimensional spaces

4.1 Case a: Real vector space

In four dimensions we can find an interpolating basis for real vector spaces. In this case equations (12) reduce to the following three equations:

$$\begin{aligned}
a_0^2 + a_1^2 + a_2^2 + a_3^2 &= 1 \\
a_0 a_2 + a_1 a_3 &= 0 \\
(a_0 + a_2)(a_1 + a_3) &= 0.
\end{aligned} \tag{21}$$

A solution (not the only solution) of these equations is provided by

$$a_0 = \frac{1}{2} \cos \theta, \quad a_1 = \frac{1}{2}(1 + \sin \theta), \quad a_2 = \frac{-1}{2} \cos \theta, \quad a_3 = \frac{1}{2}(1 - \sin \theta), \tag{22}$$

leading to the state $|\psi_{00}\rangle$,

$$|\psi_{00}\rangle = \frac{1}{2} \cos \theta |00\rangle + \frac{1}{2}(1 + \sin \theta) |11\rangle - \frac{1}{2} \cos \theta |22\rangle + \frac{1}{2}(1 - \sin \theta) |33\rangle. \tag{23}$$

The basis thus constructed is a separable state for $\theta = \frac{\pi}{2}$ and a maximally entangled for $\theta = 0$. Figure (2) shows the entanglement of the basis as a function of θ .

4.2 Case b: Complex vector space

In this case equations (12) reduce to the three independent equations

$$\begin{aligned}
|a_0|^2 + |a_1|^2 + |a_2|^2 + |a_3|^2 &= 1 \\
\overline{a_0} a_1 + \overline{a_1} a_2 + \overline{a_2} a_3 + \overline{a_3} a_0 &= 0
\end{aligned}$$

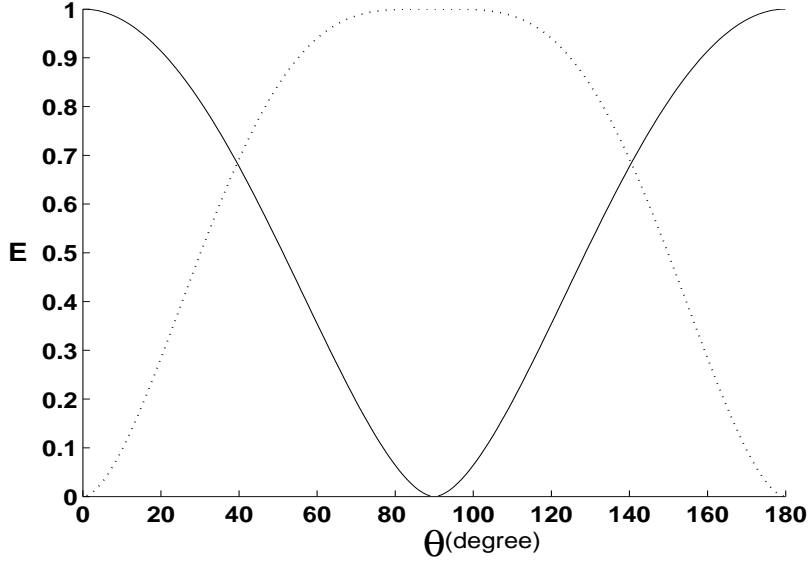


Figure 2: a) The entanglement of the basis (23) as a function of θ (in degree) (—). b) The entanglement of the basis (25) as a function of θ (in degree) (....). The entanglement is dimensionless.

$$\overline{a_0}a_2 + \overline{a_1}a_3 + \overline{a_2}a_0 + \overline{a_3}a_1 = 0. \quad (24)$$

We present one of the solutions of this set, (for more general solutions see next section). Let

$$\begin{aligned} a_0 &= \frac{1}{2}(1 + e^{i\theta} \cos \theta) \\ a_1 &= ia_2 = -a_3 = \frac{1}{2}e^{i\theta} \sin \theta. \end{aligned} \quad (25)$$

The corresponding basis interpolates between a separable basis (for $\theta = 0$) where

$$|\psi_{00}(\theta = 0)\rangle = |00\rangle \quad (26)$$

to a maximally entangled basis (for $\theta = \frac{\pi}{2}$), where

$$|\psi_{00}(\theta = \frac{\pi}{2})\rangle = \frac{1}{2}(|00\rangle + i|11\rangle + |22\rangle - i|33\rangle). \quad (27)$$

A simple calculation shows that the entanglement of this basis is equal to

$$E = -\lambda_0 \log_4 \lambda_0 - 3\lambda_1 \log_4 \lambda_1, \quad (28)$$

where $\lambda_0 = \frac{1}{4}(1 + 3\cos^2 \theta)$ and $\lambda_1 = \frac{1}{4}\sin^2 \theta$. Figure (2) shows the value of entanglement as a function of the parameter θ . It is seen that for a rather broad range of the parameter, the state is nearly maximally entangled.

Until now we have restricted ourselves to low-dimensional spaces. In the next section we provide a general solution of d dimensional spaces.

	θ_0^0	θ_1^0	θ_2^0	θ_3^0	θ_4^0	a_0	a_1	a_2	a_3	a_4
d=2	0	$\frac{\pi}{2}$	-	-	-	$\frac{1}{\sqrt{2}}$	$\frac{i}{\sqrt{2}}$	-	-	-
d=3	0	0	$\frac{2\pi}{3}$	-	-	$\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	$\frac{-i}{\sqrt{3}}$	-	-
d=4	0	0	0	π	-	$\frac{1}{2}$	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{-i}{2}$	-
d=4	0	π	π	π	-	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	-
d=5	0	$\frac{2\pi}{5}$	0	$\frac{4\pi}{5}$	$\frac{4\pi}{5}$	$\frac{1}{\sqrt{5}}e^{\frac{2i\pi}{5}}$	$\frac{1}{\sqrt{5}}e^{\frac{2i\pi}{5}}$	$\frac{1}{\sqrt{5}}e^{-\frac{2i\pi}{5}}$	$\frac{1}{\sqrt{5}}$	$\frac{1}{\sqrt{5}}e^{-\frac{2i\pi}{5}}$

Table 1: The phases θ_α^0 and the coefficients a_k 's for two,three, four and five dimensional spaces.In each case we have used the freedom to multiply all the coefficients a_k by a phase.

5 Equi-entangled bases for d dimensional spaces

In order to solve equations (12) for spaces of arbitrary dimensions we write them in matrix form as follows:

$$A^\dagger S^m A = \delta_{m,0} \quad m = 0, 1, 2, \dots, d-1, \quad (29)$$

where A is the vector of coefficients $A := (a_0, a_1, \dots, a_{d-1})^T$. In order to solve these later equations we expand the vector A in terms of the eigenvectors of S . The matrix S having the property $S^d = I$, has the following spectrum

$$S v_\alpha = \xi^\alpha v_\alpha, \quad (30)$$

where $\xi = e^{\frac{2\pi i}{d}}$ and

$$(v_\alpha)_j := \frac{1}{\sqrt{d}} \xi^{\alpha j}. \quad (31)$$

Inserting the expansion of the vector A in terms of the eigenvectors of S as

$$A = \sum_{\alpha=0}^{d-1} c_\alpha v_\alpha, \quad (32)$$

into (29), we find a system of linear equations for the absolute square of the coefficients $|c_\alpha|^2$, given by

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \xi & \xi^2 & \cdots & \xi^{d-1} \\ 1 & \xi^2 & \xi^4 & \cdots & \xi^{2(d-1)} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & \xi^{d-1} & \xi^{2(d-1)} & \cdots & \xi^{(d-1)^2} \end{pmatrix} \begin{pmatrix} |c_0|^2 \\ |c_1|^2 \\ |c_2|^2 \\ \vdots \\ |c_{d-1}|^2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \quad (33)$$

The unique solution of this last equation is given by

$$|c_0|^2 = |c_1|^2 = \cdots = |c_{d-1}|^2 = \frac{1}{d}, \quad (34)$$

or

$$c_\alpha = \frac{1}{\sqrt{d}} e^{i\theta_\alpha}, \quad \alpha = 0, 1, \dots, d-1, \quad (35)$$

where θ_α are d free parameters. Of these, one parameter can be absorbed into the total phase of state and we remain with $d-1$ continuous parameters. Inserting these values in the expansion of A , we find the final form of the desired coefficients,

$$a_k := \frac{1}{d} \sum_{\alpha=0}^{d-1} e^{i\theta_\alpha} \xi^{k\alpha}. \quad (36)$$

Thus we have found a $d-1$ parameter family of d^2 orthonormal states all of which have the same value of entanglement. When all the parameters $\theta_\alpha = 0$, we find from (36) that $a_k = \delta_{k,0}$ and hence $|\psi_{0,0}\rangle$ and all the other states become separable. By varying the parameters θ_α , we can continuously change the entanglement of the basis. The question of whether one can actually reach maximally entangled states in this family amounts to finding solutions for θ_α in the system of equations (36) such that the left hand sides are all $\frac{1}{\sqrt{d}}$ pure phases. This question is difficult to answer in its generality.

We have tried to find solutions in low dimensional spaces, although we would like to stress that the very existence of such solutions in arbitrary dimensions is not certain to us. We present some of the solutions below in table (1). Having such solutions which we denote by θ_α^0 one can easily define an interpolating basis by writing $\theta_\alpha = t\theta_\alpha^0$, where $t \in [0, 1]$.

6 Discussion

We have posed a question on the existence of a basis for a tensor product of two d dimensional spaces (qudits) so that by changing some parameters, this basis while being orthonormal, changes continuously from a product basis into a maximally entangled basis. We have also required that all the states of such a basis have the same value of entanglement during the interpolation. We call such a basis an equi-entangled orthonormal basis. We have found such bases explicitly in three and four and five dimensional spaces and have given a general solution for spaces of arbitrary dimensions. Such bases are of theoretical and possibly experimental interest for several reasons. First they provide us with a generalized Bell state measurement in which the output states are not necessarily maximally entangled but states with a prescribed value of entanglement. This application may be interesting in view of the recent results that in dimensions higher than two, maximally entangled states are not necessarily the states which have maximal non-locality. Second they provide us with a tool for

analyzing the effect of entanglement in the capacity of quantum channels, since we can encode the input alphabet into mutually distinguishable entangled codewords. Finally they may be of use in quantum cryptography with d level states.

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